

Chapter 3

Reversible Markov Chains

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Chapter 2 reviewed some aspects of the elementary theory of general finite irreducible Markov chains. In this chapter we specialize to reversible chains, treating the discrete-time and continuous-time cases in parallel. After Section 3 we shall assume that we are dealing with reversible chains without continually repeating this assumption, and shall instead explicitly say “general” to mean not necessarily reversible. 9/10/99 version

1 Introduction

Recall \mathbf{P} denotes the transition matrix and π the stationary distribution of a finite irreducible discrete-time chain (X_t) . Call the chain *reversible* if

$$\pi_i p_{ij} = \pi_j p_{ji} \text{ for all } i, j. \quad (1)$$

Equivalently, suppose (for given irreducible \mathbf{P}) that π is a probability distribution satisfying (1). Then π is the unique stationary distribution and the chain is reversible. This is true because (1), sometimes called the *detailed balance equations*, implies

$$\sum_i \pi_i p_{ij} = \pi_j \sum_i p_{ji} = \pi_j \text{ for all } j$$

and therefore π satisfies the *balance equations* of (1) in Chapter 2. 9/10/99 version

The name *reversible* comes from the following fact. If (X_t) is the stationary chain, that is, if X_0 has distribution π , then

$$(X_0, X_1, \dots, X_t) \stackrel{d}{=} (X_t, X_{t-1}, \dots, X_0).$$

More vividly, given a movie of the chain run forwards and the same movie run backwards, you cannot tell which is which.

It is elementary that the same symmetry property (1) holds for the t -step transition matrix \mathbf{P}^t :

$$\pi_i p_{i,j}^{(t)} = \pi_j p_{j,i}^{(t)}$$

and thence for the matrix \mathbf{Z} of (6) in Chapter 2:

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$$\pi_i Z_{ij} = \pi_j Z_{ji}. \quad (2)$$

But beware that the symmetry property does not work for mean hitting times: the assertion

$$\pi_i E_i T_j = \pi_j E_j T_i$$

is definitely *false* in general (see the Notes for one intuitive explanation). See Chapter 7 for further discussion. The following general lemma will be useful there.

1/31/94 version

The lemma has been copied here from Section 1.2 of Chapter 7 (1/31/94 version); reminder: it still needs to be deleted there!

Lemma 1 *For an irreducible reversible chain, the following are equivalent.*

- (a) $P_i(X_t = i) = P_j(X_t = j), \quad i, j \in I, \quad t \geq 1$
- (b) $P_i(T_j = t) = P_j(T_i = t), \quad i, j \in I, \quad t \geq 1.$

Proof. In either case the stationary distribution is uniform—under (a) by letting $t \rightarrow \infty$, and under (b) by taking $t = 1$, implying $p_{ij} \equiv p_{ji}$. So by reversibility $P_i(X_t = j) = P_j(X_t = i)$ for $i \neq j$ and $t \geq 1$. But recall from Chapter 2 Lemma 25 that the generating functions

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$$G_{ij}(z) := \sum_t P_i(X_t = j) z^t, \quad F_{ij}(z) := \sum_t P_i(T_t = j) z^t$$

satisfy

$$F_{ij} = G_{ij}/G_{jj}. \quad (3)$$

For $i \neq j$ we have seen that $G_{ij} = G_{ji}$, and hence by (3)

$$F_{ij} = F_{ji} \quad \text{iff} \quad G_{jj} = G_{ii},$$

which is the assertion of Lemma 1. ■

The discussion above extends to continuous time with only notational changes, e.g., the detailed balance equation (1) becomes

$$\pi_i q_{ij} = \pi_j q_{ji} \quad \text{for all } i, j. \quad (4)$$